Section O-0 — Preliminaries and Operator Domain Structure

Section O-0 establishes the mathematical environment in which CUWF operators live. Because CUWF operators act on entropic fields defined over the hybrid manifold

$$H = C \times M\_DOF,$$

the theory must first specify the functional setting in which those operators are meaningful. In particular, it is necessary to define the function spaces, metric-dependent norms and inner products, operator domains, and nonlinear composition rules that support the CUWF operator system.

This section therefore provides the foundation required before constructing the full CUWF Operator Algebra developed in Sections O-1 through O-6.

O-0.1 Function Spaces for CUWF Operators

A CUWF field is a scalar function

$$E : C \times M\_DOF \longrightarrow R$$

with mixed smoothness requirements across the configuration manifold and the internal degree-of-freedom manifold.

(1) Spatial domain C

On the spatial or configuration domain C, we assume

$$E(\cdot, DOF) \in W\_loc^{2,2}(C),$$

so that Δ_E and ∇E are well-defined almost everywhere.

(2) DOF domain M_{DOF}

Along the internal degree-of-freedom manifold, we require

$$E(x, \cdot) \in C^1(M_{DOF}),$$

so that the fiber derivative operator D_{DOF} is well-defined.

(3) Hybrid function space

The full CUWF function space is therefore

$$F_{CUWF} \equiv \{ f \in W_{loc}^{2,2}(C) \cap C^1(M_{DOF}) \}.$$

This hybrid functional requirement is minimal. It allows CUWF operators to be defined without imposing additional geometric or Hilbert-space structure beyond what is required by the entropic field itself.

O-0.2 Metric-Dependent Norms and Inner Product

All operator actions in CUWF depend on the entropic metric g_E. The corresponding metric-dependent norm is defined by

$$||f||_{g_E}^2 = \int f^2 |g_E| \, dx.$$

The associated inner product is

$$\langle f, g \rangle_{g_E} = \int f g |g_E| \, dx.$$

This structure has several important consequences:

The metric depends on E, ∇E, and $\text{Hess}(E)$.

The norm is therefore state-dependent.

The operator adjoints introduced in O-3 are also state-dependent.

The domains of operators may vary as the metric evolves.

This is the first core difference between CUWF operator theory and standard Hilbert-space operator theory. In CUWF, the metric is not a fixed background object; it is generated by the evolving entropic field.

O-0.3 CUWF Operators Acting on E

Three primitive operators generate the CUWF operator algebra.

(1) Entropic Laplacian $A = \Delta_E$

The first primitive operator is the entropic Laplacian, a second-order differential operator with metric coefficients:

$$Af = g_E^{ij} \nabla_i \nabla_j f.$$

Its domain is

$$D(A) = W_{loc}^{2,2}(C).$$

(2) Gradient-squared operator $B = |\nabla E|^2$

The second primitive operator is the nonlinear multiplicative gradient-squared operator:

$$B(f) = |\nabla E|^2 f.$$

Its domain is

$$D(B) = W_{loc}^{1,2}(C).$$

(3) DOF-fiber derivative operator C = D_DOF

The third primitive operator is the derivative along the internal degree-of-freedom manifold:

$$C(f) = D_{DOF} f.$$

Its domain is $D(C) = C^1(M_{DOF})$.

(4) Mixed compositions

Operators such as

$$A \circ B, \quad B \circ C, \quad A \circ C \circ B,$$

and higher-order compositions arise naturally in CUWF. These compositions are nonlinear and depend on the evolving entropic field E. They are not merely formal products; they encode the coupling among curvature smoothing, gradient sharpening, and internal DOF drift.

O-0.4 Nonlinear Operator Setting

The CUWF Master Operator is

$$\mathfrak{E}_E = A - \alpha B + \beta C.$$

This operator is nonlinear in several distinct senses.

(1) Nonlinear in f

The operators B and C act multiplicatively through E and its derivatives, so their action cannot be reduced to ordinary linear transformation on a fixed vector space.

(2) Nonlinear in operator composition

The composition X ∘ Y is not bilinear in the conventional algebraic sense because the metric and the operators themselves depend on the evolving entropic field.

(3) Metric-dependent

The coefficients of A depend on g_E(E), and therefore change when E changes.

(4) Adjoint-dependent nonlinearity

Adjoints vary dynamically with E, because the inner product itself is metric-dependent.

Consequently, the algebra generated by {A, B, C} is:

non-Lie;

non-C*;

non-polynomial;

non-closed, generating an infinite operator tower; and

state-dependent.

This justifies the need for a dedicated CUWF Operator Algebra, developed from Section O-1 onward.

O-0.5 Summary of Section Purpose

Section O-0 establishes the mathematical foundation required for CUWF operator theory. It defines:

the hybrid function space for CUWF fields;

metric-dependent norms and inner products;

the domains of the fundamental operators;

the nonlinear structure of operator composition; and

the state-dependence of algebraic and adjoint relations.

This provides the mathematical foundation required for:

O-1 — decomposition of $\mathfrak{E}$_E;

O-2 — operator commutator algebra;

O-3 — adjoint theory;

O-4 — nonlinear spectrum theory;

O-5 — operator evolution; and

O-6 — nonlinear quantization framework.